THE EIGENVECTORS OF THE RIGHT–JUSTIFIED PASCAL TRIANGLE: A SHORTER PROOF WITH GENERATING FUNCTIONS

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Let $R = \left(\binom{i-1}{n-j}\right)_{1\le i,j\le n}$, $a = \frac{1+\sqrt{5}}{2}$, $\lambda_j = (-1)^{n+j}a^{2j-n-1}$, $1 \le j \le n$,

$$u_{ij} = \sum_{k=1}^{j}(-1)^{i-k}\binom{i-1}{k-1}\binom{n-i}{j-k}a^{2k-i-1}$$

and $\mathbf{u}_j = (u_{ij})_{1\le i\le n}$.

In [1] Callan proves that $R\mathbf{u}_j = \lambda_j\mathbf{u}_j$ for $1 \le j \le n$ by what he calls a *bracing exercise in manipulating binomial coefficient sums.* Here we give a generating function approach that might be a bit simpler.

We need also the quantity $b = \frac{1-\sqrt{5}}{2}$.

Consider the generating function $U_i(z) = z(1+z)^{n-i}(az+b)^{i-1}$. It is immediate that

$$U_i(z) = \sum_{j=1}^{n} u_{ij}z^j.$$

We must prove that $(R\mathbf{u}_j)_i = (\lambda_j\mathbf{u}_j)_i$ for all i with $1 \le i \le n$. Now

$$\begin{aligned}
(R\mathbf{u}_j)_i &= [z^j]\sum_{k=1}^{n} R_{ik}U_k(z)\\
&= [z^j]\sum_{k=1}^{n}\binom{i-1}{n-k}z(1+z)^{n-k}(az+b)^{k-1}\\
&= [z^{j-1}](az+b)^{n-i}\sum_{k\ge0}\binom{i-1}{k}(1+z)^k(az+b)^{i-k-1}\\
&= [z^{j-1}](az+b)^{n-i}\big(1+b+z(1+a)\big)^{i-1}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(\lambda_j\mathbf{u}_j)_i &= [z^j]\sum_{j=1}^{n}(-1)^{n+j}a^{2j-n-1}u_{ij}z^j\\
&= [z^j](-1)^n a^{-n-1}\sum_{j=1}^{n}u_{ij}(-za^2)^j
\end{aligned}$$

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\begin{aligned}
&= [z^j](-1)^n a^{-n-1} U_i(-za^2) \\
&= [z^j](-1)^n a^{-n-1}(-za^2)(1-za^2)^{n-i}(b-a^3z)^{i-1} \\
&= [z^{j-1}](za-\tfrac{1}{a})^{n-i}(a^2z-\tfrac{b}{a})^{i-1},
\end{aligned}
$$

and the claim follows since $b=-\frac{1}{a}$, $1+a=a^2$, and $-\frac{b}{a}=1+b$.

References

1. D. Callan, *The eigenvectors of the right–justified Pascal triangle*, arXiv:math.CO/0011081, (2000), 5 pages.

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